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Derivation of the Most General Relativistic Transformation Law of Quantum Fields

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In this note we derive the most general relativistic transformation law of quantum fields from the Relativity Principle. In the special case of frame-independent fields it reduces to the familiar forms [compare equation (8)] of the current QFT ($\equiv$ quantum field theory), due originally to Wigner. This is a modified, and perhaps simpler, version of that reported in a previous paper (Ingraham, 1962).

Just a word of motivation. This more general relativistic formalism is not just a mathematical curiosity, but should be of vital interest because it allows the incorporation of a cut-off† into quantum fields and thus into the S-matrix, while the conventional formalism does not (the latter is not proven, but forty years of vain effort in field theory teaches us this). In other words, we suggest that the unnecessarily restrictive form of relativistic invariance currently used is mainly responsible for QFT's well-known pathology.

Start with the set of equivalent 'observers', or frames, $\mathscr{L}$, $\mathscr{L}'$, $\mathscr{L}''$,.... $\mathscr{L}$ stands for a space-time frame and comprises also the 'coordinate frame', or basis, in state vector Hilbert Space $\mathscr{H}$ when we are talking about QFT. *Equivalent* means that the theory should not prefer one to another, that they are *intrinsically* indistinguishable. Consider the orthonormal basis‡ $|0\rangle$, $|\mathbf{k}\rangle$, $|\mathbf{k}_1,\mathbf{k}_2\rangle$, etc. where $|0\rangle$ is the no-meson state (vacuum), $|\mathbf{k}\rangle$ is a free incoming one-meson state with 3-momentum $\mathbf{k}$ relative to $\mathscr{L}$'s axes, $|\mathbf{k}_1,\mathbf{k}_2\rangle$ is an incoming two-meson state with 3-momenta $\mathbf{k}_1$, $\mathbf{k}_2$ relative to $\mathscr{L}$, etc. Similarly let $|0\rangle'$ $(=|0\rangle)$, $|\mathbf{k}'\rangle'$, $|\mathbf{k}_1{}',\mathbf{k}_2{}'\rangle'$,.. denote an orthonormal basis for any other frame $\mathscr{L}'$, where $\mathbf{k}'$, etc. are the free meson 3-momenta relative to frame $\mathscr{L}'$. Consider the state $|\mathbf{k}\rangle'$, where $\mathbf{k}$ are the same three numbers that appear in $|\mathbf{k}\rangle$: these states are called *subjectively identical* for frames $\mathscr{L}'$ and $\mathscr{L}$, with the corresponding definition for two and higher-meson states. $|\mathbf{k}\rangle'$ and $|\mathbf{k}\rangle$ are different ($\equiv$objectively different) states of course, since if $\mathscr{L}$ sees a meson moving in the direction $\mathbf{k}$, then these same three numbers for $\mathscr{L}'$ define a (in general) different direction. But the name is justified because the state $|\mathbf{k}\rangle'$ 'looks the same' to $\mathscr{L}'$ as the state $|\mathbf{k}\rangle$ does to $\mathscr{L}$.

† See section entitled ' Frame-dependence and Cut-offs ', p. 86.

‡ These are the usual state-densities, normalized to δ-functions, properly speaking.

Let $U(L)$ connect subjectively identical states of the two bases:

$$\begin{aligned} |0\rangle' &= |0\rangle = U(L)|0\rangle \\ |\mathbf{k}\rangle' &= U(L)|\mathbf{k}\rangle \\ |\mathbf{k}_1, \mathbf{k}_2\rangle' &= U(L)|\mathbf{k}_1, \mathbf{k}_2\rangle, \text{ etc.} \end{aligned} \tag{1}$$

where L is the Poincaré group P element connecting $\mathscr{L}$ and $\mathscr{L}'$'s coordinates: $x' = Lx \equiv \Lambda x + a$. $U(L)$ transforms an orthonormal basis into another such and is therefore unitary. This is the *definition* of the representation $U(L)$ of P on state vector Hilbert Space adopted by us—it is the same definition used in current QFT. [Note that we can write mathematically $|\mathbf{k}\rangle = a^*(\mathbf{k})|0\rangle$, $|\mathbf{k}\rangle' = a'^*(\mathbf{k})|0\rangle'$, etc., hence write equivalently to (1) $a'(\mathbf{k}) = U(L)a(\mathbf{k})U(L)^{-1}$ and the same thing for the '4-dimensional' (covariant) operators $a'(k)$ and $a(k)$.]

Consider now a scalar ($\equiv$spinless) physical field (the 'meson') for mathematical simplicity. A quantum field theory should require that each frame $\mathscr{L}$ disposes of a one (space-time)-component operator-valued field: $\mathscr{L} \to \phi(\mathscr{L})$, $\mathscr{L}' \to \phi(\mathscr{L}')$, etc. N.B., there is nothing in either the Relativity Principle or general physical theory which forces us to assume that the 'meson' must be represented by the *same* mathematical field for all observers† (which can be referred to any frame of course), therefore for the time being we keep full generality by allowing $\phi(\mathscr{L})$ and $\phi(\mathscr{L}')$ to be the (possibly different) mathematical fields representing the 'meson' for frames $\mathscr{L}$ and $\mathscr{L}'$ respectively. $\phi(x;\mathscr{L})$ is $\mathscr{L}$'s field referred to his own frame, and $\phi'(x';\mathscr{L}')$ is $\mathscr{L}'$'s field referred to his own frame, 'frame' meaning both in Minkowski Space and Hilbert Space, we recall. $\phi(\mathscr{L}')$ can be referred to frame $\mathscr{L}$, in which case it is written $\phi(x;\mathscr{L}')$. It is a scalar, thus by definition $\phi(x;\mathscr{L}') = \phi'(x';\mathscr{L}')$. $\phi(\mathscr{L})$ and $\phi(\mathscr{L}')$ would be *different* if, when referred to any common frame ($\mathscr{L}$ for example) they were different operator-valued functions of x:

$$\phi(x;\mathscr{L}') \neq \phi(x;\mathscr{L}) \tag{2}$$

Conventional QFT is included in this formalism as the special case $\phi(x;\mathscr{L}') = \phi(x;\mathscr{L}) \equiv \phi(x)$, any two $\mathscr{L}$, $\mathscr{L}'$.

So having chosen the mathematical formalism, namely an ensemble $\{\phi(\mathscr{L}), \phi(\mathscr{L}'), \phi(\mathscr{L}''), \ldots\}$, one for each Lorentz frame, of mathematical fields to represent the physical 'meson', we turn to the question of its transformation law under P. Namely, how is the Relativity Principle $\equiv$ complete equivalence of $\mathscr{L}$, $\mathscr{L}'$, $\mathscr{L}''$,... expressed in this formalism?

Let us form certain observable c-numbers from the operator fields $\phi(\mathscr{L})$ and $\phi(\mathscr{L}')$ in order to formulate the Relativity Principle. If $|n\rangle$ is an

† It seems to us that this possibility was overlooked in the past for purely semantic reasons: '*the* (physical) π-meson field', being singular in the linguistic structure, inevitably forced the association of a unique mathematical field $\phi(x)$, '*the* (mathematical) π-meson field', to it.

n-particle state of $\mathscr{L}$'s basis, n standing for a particular set of n-momenta, and similarly for $|n'\rangle'$, consider the c-numbers

$$\langle m|\phi(x;\mathscr{L})|n\rangle, \qquad '\langle m'|\phi'(x';\mathscr{L}')|n'\rangle' \tag{3}$$

These represent expectation values or more generally transition matrix elements, in principle all observable. In this notation $|n\rangle'$ is the element of $\mathscr{L}'$'s basis subjectively identical to $|n\rangle$, i.e., n stands for one and the same set of numbers in the two cases. Similarly two events P' and P will be called subjectively identical if they have numerically the same coordinates relative to frames $\mathscr{L}'$ and $\mathscr{L}$ respectively: $x' = x$.

Now we can put the Relativity Principle verbally this way:

> Relativity Principle: if two equivalent observers do the same ($\equiv$subjectively the same!) experiment, they must get identical numbers (4)

Translating this into the language of the matrix elements (3), this means that if $n' = n$, $m' = m$, and $x' = x$ (i.e., the subjective identity of the corresponding states and of the two events) then

$$'\langle m|\phi'(x;\mathscr{L}')|n\rangle' = \langle m|\phi(x;\mathscr{L})|n\rangle, \qquad \text{all } m, n, x \tag{5}$$

But now we can use (1) to write $|n\rangle' = U(L)|n\rangle$ and $'\langle m| = \langle m|U(L)^{-1}$. Then since the $\{|n\rangle\}$ and $\{|m\rangle\}$ are complete we can 'cross them out' in (5) to infer:

Relativistic invariance (in passive form)

$$\phi'(x;\mathscr{L}') = U(L)\,\phi(x;\mathscr{L})\,U(L)^{-1}, \qquad \mathscr{L}' = L^{-1}\mathscr{L} \tag{6}$$

where the notation $\mathscr{L}' = L^{-1}\mathscr{L}$ is explained† in a previous paper (Ingraham, 1962) and many places elsewhere in our work. We can put this into 'active form' ($\equiv$only components of both fields relative to one frame, say $\mathscr{L}$, occurring) by using the scalarity of $\phi(\mathscr{L}')$ which implies $\phi'(x;\mathscr{L}') = \phi(L^{-1}x;\mathscr{L}')$. Thus:

Relativistic invariance (in active form)

$$\phi(L^{-1}x;\mathscr{L}') = U(L)\,\phi(x;\mathscr{L})\,U(L)^{-1}, \qquad \mathscr{L}' = L^{-1}\mathscr{L} \tag{7}$$

Equations (6) or (7) are the sought for relativistic transformation law of frame-dependent quantum fields. In the special case of no frame-dependence, (7), for example, becomes

$$\phi(L^{-1}x) = U(L)\,\phi(x)\,U(L)^{-1} \tag{8}$$

the familiar law of present-day QFT.

† It is sufficient to say that $\mathscr{L}' = L^{-1}\mathscr{L}$ is equivalent to $x' = Lx \equiv \Lambda x + a$. The notation is suggested by the fact that if $n(\mathscr{L})$ is the unit time-like vector aligned along $\mathscr{L}$'s positive time axis, and similarly for $n(\mathscr{L}')$, then

$$n(\mathscr{L}')^{\mu} = \Lambda^{-1\mu}{}_{\nu}(\mathscr{L})^{\nu}$$

where (N.B.!) the unprimed indices refer to frame $\mathscr{L}$ [thus $n(\mathscr{L})^{\nu} = (0001)$].

Frame-dependence and Cut-offs

In view of wide-spread misunderstanding of this proposal, we wish to supplement the precise formulation above with some further qualitative remarks.

This more general transformation law (6) or (7) is no way an abandonment of strict relativistic invariance. That is, *there is no preferred frame, or frames; all inertial frames are still strictly equivalent.* What do we mean by that? Simply that any two observers, if they do corresponding experiments, get exactly the same numbers. [The precise formulation is definition (4), above, and its verification, equation (5).] Not even the strictest exponent of the *status quo* could quarrel with this as the basic meaning of 'relativistic invariance', nor argue that this is a relaxation or abandonment of the same. It is profoundly different from nonrelativistic theories with cut-offs. There, one particular frame is preferred, therefore giving different numbers for experiments performed in this frame, as against those performed in others.

Now using the time-like unit vector n one can introduce the *spatial* momentum squared (relative to n)

$$k_\perp{}^2 \equiv k^2 + (n.k)^2 \geqslant 0$$

which is positive for any k, and exploiting this, achieve a cut-off in fields and S-operator consistent with all general demands, as we have shown elsewhere.† Frame-dependence is introduced through explicit dependence on n, which can be identified with a frame, its rest frame. The fallacious counter-argument then proceeds as follows: 'You have a unit time-like vector n in your fields and S-operator. But this certainly prefers a frame, namely the rest frame, in which the components n^μ are (0001). Therefore relativistic invariance is violated, Q.E.D.'

The fallacy enters via the phrase '*a* unit time-like vector'. In a theory satisfying the general relativity criterion (6) or (7), not one such vector, but all unit time-like vectors must enter. They can be identified with Lorentz frames $\mathscr{L}$ as explained, and labelled $n(\mathscr{L})$. Since *all* $n(\mathscr{L})$ enter, none is preferred. We give an example to clarify this.

Observer $\mathscr{L}$ does a scattering experiment with incoming and outgoing particles of momenta $p_1, p_2, \ldots$ and $p_1', p_2', \ldots$ respectively. His S-matrix element, nonsingular because of the cut-off made possible by the $n(\mathscr{L})$-dependence, is a function of the numbers $p_1, p_2, \ldots; p_1', p_2', \ldots$ and $(0001) \equiv n(\mathscr{L})^\mu$, $\mathscr{L}$'s unit normal *referred to his own frame.* Equivalent observer $\mathscr{L}'$ does the 'corresponding' experiment, i.e., with momenta $p_1, p_2, \ldots$ and $p_1', p_2', \ldots$, the same numbers, but now meaning momenta referred to his frame. In his matrix element he uses his normal $n(\mathscr{L}')$ referred to his own frame, namely $n(\mathscr{L}')^{\mu'} = (0001)$. Thus he gets numerically the same S-matrix element. N.B. $\mathscr{L}'$ does *not* use $n(\mathscr{L})$, referred to his frame, namely $n(\mathscr{L})^{\mu'} \neq (0001)$, which would introduce a preferred frame and breakdown of equivalence.

† See the list of references given in Ingraham, R. L. (1967). *Renormalization Theory of Quantum Field Theory with a Cut-off*, Chapter 13. Gordon and Breach, London.

Thus there is the possibility of well-defined S-matrix elements and at the same time no relaxation of strict equivalence of Lorentz frames.

This freedom has been bought at the expense of allowing *not one* but a multiplicity of fields $\phi(\mathscr{L})$ and S-operators $S(\mathscr{L})$. They transform among themselves according to (7) and

$$S(L^{-1}\mathscr{L}) = U(L)\,S(\mathscr{L})\,U(L)^{-1} \tag{9}$$

which follows from it. The reader who has followed the argument this far may now wonder whether this multiplicity of fields and S-operators does not immediately contradict experiment. The fact is that an experiment which could decide this has never been performed.

The $n(\mathscr{L})$-dependence will enter multiplied by a small length λ (the cut-off), thus will be slight. So very high energies would be necessary in any case. But finding evidence of a cut-off in a high-energy scattering experiment would not be conclusive, since the same effect could conceivably be produced by a frame-independent cut-off (if that were possible, which we do not believe). What is needed is a comparison of scattering *performed* in relatively moving frames at the same values of s and t and transformed to a common frame. For example, elastic scattering performed in both the CM and Lab frames of a pair of particles, and the Lab frame cross-section transformed to the CM. For then the two S-matrix elements would involve the different numbers $n(\mathscr{L}_{\text{CM}})^\mu = (0001)$ and $n(\mathscr{L}_{\text{Lab}})^\mu \neq (0001)$, where the index μ refers to CM frame components.

Numerical calculations for pure quantum electrodynamical processes (ee and e^+e^- elastic scattering), where perturbation theory should be good, have been made and presented elsewhere (Ingraham, 1965).

To our knowledge, such an experiment has never been done. Note that for ee and e^+e^- scattering, the Lab energy would have to be as high as possible to show up the presence of the frame-dependence (probably $p_{\text{Lab}} \gtrsim 10$ GeV/c), while because of the smallness of the electron mass, the corresponding CM energy is quite low: $p_{\text{CM}} \sim 50$ MeV/c.

References

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